Pergamon

J. Appl. Maths Mechs, Vol. 62, No. 4, pp. 605–616, 1998

0021–8928/98/$—see front matter

PII: S0021–8928(98)00077-X

THE ERRORS DUE TO APPROXIMATING UNBOUNDED ELASTIC BODIES BY BOUNDED ONES†

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(*Received* 13 *November* 1996)

The stress–strain state of an isotropic plane, weakened by a cavity of arbitrary shape, is sought approximately by solving the elastic problem in a large circle with the same cavity. An asymptotic analysis of the problem is carried out with a parameter (the radius of the circle) and, among the variety of stable natural and artificial boundary conditions on the circle, the one is chosen which provides the best approximation of the problem for an infinite body. Asymptotically accurate estimates of the errors in calculating the displacements, stresses and their derivatives are given. Possible extensions of this approach to other bodies, including three-dimensional or two-dimensional wedge-shaped bodies are discussed.

1. FORMULATION OF THE PROBLEMS

Consider an isotropic homogeneous elastic plane $\mathbf{R}^n$ (a space with $n = 3$), which is weakened by a cavity ω containing the origin of coordinates $x = 0$. The displacement vector $u = (u_1, \ldots, u_n)$ satisfies the system of Lamé equations

$$L(\nabla)u(x) \equiv -\mu\nabla\cdot\nabla u(x) - (\lambda+\mu)\nabla\nabla\cdot u(x) = f(x), \quad x \in \Omega = \mathbf{R}^n \setminus \overline{\omega} \tag{1.1}$$

Here λ and μ are the Lamé coefficients, f is the vector of the mass forces, ∇ = grad, $\nabla\cdot$ = div and $\nabla\cdot\nabla = \Delta$ is the Laplace operator. We set the following boundary conditions on $\partial\omega$

$$N(x, \nabla)u(x) \equiv \sigma^{(\nu)}(u;\ x) = g(x), \quad x \in \partial\omega \tag{1.2}$$

$$u(x) = g(x), \quad x \in \partial\omega \tag{1.3}$$

In (1.3) g is the vector of forced displacements and in (1.2) g is the vector of the external loads, and ν is the unit vector of the inward normal (with respect to ω) to the surface $\partial\omega = \partial\Omega$, which, for simplicity, we will assume to be smooth. Moreover

$$\sigma_{ij}(u) = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda\nabla\cdot u, \quad \sigma_j^{(\nu)}(u) = \nu_i\sigma_{ij}(u)$$

Summation is carried out over repeated subscripts in the limits from 1 to n and δ_{ij} is the Kronecker delta.

Suppose $R \in [R^0, +\infty)$ and the quantity R^0 are so large that the circle (a sphere when $n = 3$) $B_{R0} = \{x \in \mathbf{R}^n : |x| < R^0\}$ contains the set $\overline{\omega} = \omega \cap \partial\omega$. Consider the equations

$$L(\nabla)u^R(x) = f(x), \quad x \in \Omega_R = B_R / \overline{\omega} \tag{1.4}$$

$$M^R(x, \nabla)u^R(x) = h^R(x), \quad x \in \partial B_R \tag{1.5}$$

supplemented by one of the boundary conditions

$$N(x, \nabla)u^R(x) = g(x), \quad x \in \partial\omega \tag{1.6}$$

$$u^R(x) = g(x), \quad x \in \partial\omega \tag{1.7}$$

The solution u^R of problem (1.4)–(1.6) in the truncated region Ω_R is interpreted as an approximation to the solution u of the external problem. The following question arises: how should the operator M^R

†*Prikl. Mat. Mekh.* Vol. 62, No. 4, pp. 650–663, 1998.

(x, ∇) of the boundary conditions on the far boundary ∂B_R be chosen so as to obtain the best rate of convergence of u^R to u as $R \to \infty$?

This problem has long attracted the attention of mathematicians in view of the fact that many numerical methods (for example, the finite-elements method) require changing from an unbounded region to a bounded region. The external three-dimensional Dirichlet problems for the Laplace and Stokes operators were discussed in [1–4], artificial boundary conditions on ∂B_R were chosen, and the difference $u - u_R$ was estimated in an energy norm. Boundary-value problems for a three-dimensional Lamé system were investigated in [5] (a brief description of the results will be given in Section 2). Three possibilities were considered: the surface ∂B_R is rigidly clamped ($M^R u = u$ are the Dirichlet conditions), is stress-free ($M^R u = Nu$ is the Neumann condition) and the elastic closing conditions (the mixed boundary conditions) are realized on ∂B_R. In the third case, the operator M_R is defined by the formula

$$M^R(x, \nabla)u^R(x) = \sigma^{(\nu)}(u^R;\ x) + A(x)u^R(x) \tag{1.8}$$

where $\nu = x|x|^{-1}$ is the outward normal to ∂B_R and A is a symmetric and non-negative definite 3×3 matrix function. It turns out that, with the appropriate choice of A, problem (1.4)–(1.6) with the operator M^R from (1.8) gives the best accuracy $O(R^{-2})$ of the approximation.

Problems of the theory of elasticity in unbounded regions are an idealization of the actual problems for bodies of finite dimensions. Nevertheless, it is precisely these problems that play a predominant role in asymptotic analysis (see [6–9], etc.), and their integral characteristics occur in various asymptotic formulae (see [7, 10–13], etc.). The most important of these characteristics are the matrices (tensors) of elastic capacity and polarization, which generalize the well-known classical objects in the theory of harmonic functions (see, for example, [14]). The elements of these matrices are expressed in terms of integrals of the solutions of problems (1.1), (1.3) or (1.1), (1.2) with special right-hand sides, i.e. for these it is sufficient to increase the accuracy of the calculations only in the region of the opening ω. In Sections 3–5 we distinguish a zone of improved approximation by means of weighted estimates of the difference $u - u^R$ and its derivatives.

The majority of methods used later are suitable for anisotropic or non-uniform materials (see Section 6), but even in the case of isotropy, an approximate determination of the polarization matrices by truncation of the regions makes sense, since simple representations of these in terms of conformal mappings in [15, 16] turn out to be invalid (the key relations (5)–(8) in [15] and (17) and (21) in [16] are erroneous, and their correction leads to awkward final formulae).

The results of this paper, relating to problem (1.1) and (1.2), remain unchanged when the cavity is replaced by an elastic inclusion.

2. THE THREE-DIMENSIONAL PROBLEM

The coordinate $x \mapsto \xi = R^{-1}x$ of the region Ω_R is transformed by compression into a unit sphere with a small cavity $\omega(R) = \{\xi \in \mathbf{R}^3 : R\xi \in \omega\}$. Hence, we must interpret problem (1.4)–(1.6) as a singular perturbation of problem (1.1), (1.2). Methods of constructing the asymptotic form of the solutions of such problems have been developed considerably (see [6, 9, 17], etc.). We will use the method of matched asymptotic expansions and we will seek a solution in the form of the formal series

$$u^R(x) = \sum_{k=0}^{\infty} R^{-k}(v^k(x) + w^k(R^{-1}x)) \tag{2.1}$$

Terms of the series v^k, which depend on the "fast" variables x, are solutions of problems (1.1), (1.2) with certain right-hand sides, defined recurrently. The vector functions w^k are written using the slow variables $\xi = R^{-1}x$, and their role consists of compensating for discrepancies of v^k left in the boundary condition (1.5) on ∂B_R. They are found when solving problems of the following type

$$L(\nabla_\xi)w(\xi) = 0, \quad \xi \in B_1 = \{\xi : |\xi| < 1\} \tag{2.2}$$

$$M(\xi, \nabla_\xi)w(\xi) = h(\xi), \quad \xi \in \partial B_1 \tag{2.3}$$

Here $L(\nabla_\xi)$ is the Lamé operator from (1.1), containing differentiation with respect to ξ, while the operator M from the boundary condition on the unit sphere ∂B_1 is related to the operator M^R from (1.5) as follows:

$$M^R(x, \nabla_x) = M^R(R\xi, R^{-1}\nabla_\xi) = R^{\varkappa} M(\xi, \nabla_\xi) \tag{2.4}$$

Here $\varkappa$ is an integer which shows the degree of generalized homogeneity of M^R with respect to the replacement $x \mapsto \xi$. Essentially (2.4) is the first of the limitations imposed on the structure of the operator of the artificial boundary conditions.

It is logical to assume that the fundamental term υ^0 in (2.1) is identical with the solution u of the external problem (1.1), (1.2). Hence, the next aim is, by choosing the operator M^R, to annul as many of the following terms of series (2.1) as possible. Thus, if we show that

$$\upsilon^0 = u, \quad w^0 = \ldots = w^{K-1} = 0, \quad \upsilon^1 = \ldots = \upsilon^{K-1} = 0 \tag{2.5}$$

then u^R is an approximation to u of order K and when $x \in \Omega_R$

$$u(x) - u^R(x) = O(R^{-K}) \tag{2.6}$$

For the complete expansion (2.1) to hold we must assume that the right-hand side f of system (1.1) is finite (equal to zero when $|x| > R_1$). However, the final formulation of the result (see later, Section 3) requires a weaker constraint

$$f(x) = O(|x|^{-4-\varepsilon}) \tag{2.7}$$

Here ε is an arbitrary (generally speaking, small) positive number, and to reduce the amount of writing we will agree that formula (2.7) can be differentiated as many times as desired, assuming that $\nabla O(|x|^\tau) = O(|x|^{\tau-1})$. As is well known (see, for example, [18, Section 6.4] and [5]), a unique solution of problem (1.1), (1.2) exists, which vanishes at infinity, which, moreover, allows of the asymptotic representation

$$u(x) = T(x)a + O(|x|^{-2}) \tag{2.8}$$

Here a is a column which depends on f and g (all the vectors are realized as columns in the Cartesian representation), while T is the Kelvin–Somigliana tensor, considered as a 3×3 matrix with elements

$$T_{ij}(x) = \alpha|x|^{-1}[(1+2\gamma)\delta_{ij} + x_i x_j |x|^{-2}]$$

$$\alpha = (\lambda+\mu)[8\pi\mu(\lambda+2\mu)]^{-1}, \quad \gamma = \mu(\lambda+\mu)^{-1}$$

Suppose initially that M^R is the operator of the Dirichlet conditions, $\varkappa = 0$ in (2.4) and

$$M^R(x, \nabla)u^R(x) \equiv u^R(x) = 0, \quad x \in \partial B_R \tag{2.9}$$

Then the principal term of the discrepancy, generated by u in boundary conditions (2.9), is equal to $T(x)a = R^{-1}T(\xi)$ and is compensated by the solution w^1 of system (2.2) with the Dirichlet condition

$$w^1(\xi) = -T(\xi)a, \ \xi \in \partial B_1$$

Thus, $w^0 = 0$, i.e. relations (2.5) with $K = 1$ are satisfied, but the term w^1 in (2.1) need not be annulled. Consequently, problem (1.4), (2.9), (1.6) yields an approximation to u of the first order.

We will investigate the case of mixed artificial boundary conditions. Suppose the elements of the 3×3 matrix are defined as

$$A_{ij}(x) = \frac{2\mu}{1+\gamma}|x|^{-1}\left(\gamma\delta_{ij} + \frac{3+5\gamma}{2(1+\gamma)}\right) \tag{2.10}$$

For the operator M^R from (1.8) relation (2.4) holds with $\varkappa = -1$. As calculations show [5], for any column a the following equality holds

$$M^R(x, \nabla)T(x)a = \sigma^{(\nu)}(Ta; x) + A(x)T(x)a = 0 \tag{2.11}$$

Hence, in view of (2.8) and (2.11) the discrepancy of the solution u in conditions (1.5) is $O(R^{-3})$. Hence, the terms w^0 and w^1 in series (2.1) disappear; moreover, the term υ^1, designed to eliminate the additional discrepancy generated by w^1 in (1.6), also drops out.

Thus, formulae (2.5) hold for $K = 2$, i.e. the solution of problem (1.4)–(1.6) with $h^R = 0$ and with the operator M^R, taken from (1.8) and (2.10), yields a second-order approximation (better than when setting conditions (2.9) on ∂B_R).

If $A = 0$ in (1.8), then M^R takes the form of the operator of the boundary conditions in the stresses. The Neumann problem thus obtained in Ω_R is solvable if six equations are satisfied (the principal vector and moment of the loads are necessarily annulled), and its solution is determined apart from rigid displacements. It has been established [5] (see also Section 4 below), that with a special choice of h^R in (1.5) the stresses $\sigma_{ij}(u^R)$ serve as a good approximation for $\sigma_{ij}(u)$ in Ω_R. Nevertheless, the operator M^R with A from (2.10) possesses obvious advantages: homogeneity ($h^R = 0$) of boundary condition (1.5) and unique solvability of the corresponding problem (1.4)–(1.6).

3. DISCUSSION

The main ideas in constructing artificial boundary conditions are particularly obvious in the example of the Laplace operator with the fundamental solution in $\mathbf{R}^3$

$$\Phi(x) = (4\pi|x|)^{-1} \tag{3.1}$$

Here the solution u of the external problem (Dirichlet or Neumann) allows of the expansion $u(x) = a_0\Phi(x) + O(|x|)^{-2})$. The operator M^R is chosen so that it cancels the principal asymptotic term $a_0\Phi$. In the scalar case considered, it is possible to achieve this cancellation on any truncating surface $\Gamma_R = \partial\Omega_R\backslash\partial\omega$ (attention was given to this in [5], but in [1, 2] Γ_R is a sphere). Thus, if $\Gamma_R = \{x: R^{-1}x \in \Gamma\}$, and Γ is a surface with outward normal ν, then

$$M^R(x, \nabla) = \nu_j \frac{\partial}{\partial x_j} + \frac{\nu_j x_j}{|x|^2} \equiv \partial_\nu + A_0(x)$$

and

$$M^R(x, \nabla)\Phi(x) = -\nu_j \frac{1}{4\pi} \frac{x_j}{|x|^3} + \frac{\nu_j x_j}{|x|^2} \frac{1}{4\pi|x|} = 0 \tag{3.2}$$

However, the single requirement (3.2) is insufficient, since the problem obtained in Ω_R must possess "good" properties. Green's formula holds, and this leads to a variational formulation of the problem

$$-\int_{\Omega_R} u\Delta u dx + \int_{\partial\omega} u\partial_\nu u ds_x + \int_{\Gamma_R} u(\partial_\nu u + A_0 u) ds_x = \tag{3.3}$$

$$= \int_{\Omega_R} |\nabla u|^2 dx + \int_{\Gamma_R} A_0 |u|^2 ds_x$$

If, moreover, $A_0(x) = \nu \cdot x > 0$ when $x \in \Gamma$ (the inequality is guaranteed, in this case, when the region bounded by Γ is star-like with respect to the point $x = 0$; see [5]), the quadratic form on the right in (3.3) is positive definite and so the problem is uniquely solvable.

The problem of the theory of elasticity (1.4)–(1.6) with the operator of the artificial boundary conditions (1.8) possesses all the above-mentioned properties. In particular, by Betti's identity, we have

$$\int_{\Omega_R} u \cdot Lu dx + \int_{\partial\omega} u \cdot \sigma^{(\nu)}(u) ds_x + \int_{\partial B_R} u \cdot (\sigma^{(\nu)}(u) + Au) ds_x =$$

$$= 2E(u; \Omega_R) + \int_{\partial B_R} u \cdot Au ds_x \equiv 2E_A(u; \Omega_R)$$

Since the functional $E(\cdot; \Omega_R)$ of the elastic energy is only degenerate on rigid displacements, the form $E_A(\cdot; \Omega_R)$ is positive definite in view of the above-mentioned properties of the matrix A from (2.10). However, the results obtained for the three-dimensional problem of the theory of elasticity is not as complete as in the case of the Laplace operator due to the requirements that the body Ω should be isotropic and Γ_R should be spherical. It has been verified [5], that for a non-spherical surface Γ_R the matrix A obtained from condition (3.2) loses the symmetry necessary for the variational formulation of problem (1.4)–(1.6). We do not know whether the truncating surfaces which provide desirable properties to the problem in Ω_R exist for anisotropic bodies.

The two-dimensional case differs considerably from the three-dimensional one. We will explain this using the example of the Laplace operator, for which the fundamental solution in a plane has the form

$$\Phi(x) = -(2\pi)^{-1} \ln|x| \tag{3.4}$$

Unlike (3.1) it increases as $x \to \infty$, which leads to an "incorrect" sign on A_0 and to a loss in solvability in the corresponding problem. In fact

$$\partial_\nu \Phi(x) + A_0(x)\Phi(x) = -\frac{1}{2\pi}\nu_j \frac{x_j}{|x|^2} - A_0(x)\frac{1}{2\pi}\ln|x|$$

and so $A_0(x) = -\nu \cdot x|x|^{-2}(\ln|x|)^{-1}$, and the inequality $A_0(x) \geqslant 0$ is impossible everywhere on Γ_R (in particular $A_0 = -(R \ln R)^{-1}$ for the neighbourhood $\Gamma_R = \partial B_R$). The following sections, devoted to the plane problem of the theory of elasticity, are therefore based on other considerations.

The discussions carried out in Section 2 have a formal character and therefore need confirmation. The approaches used previously in [1–4], give only weak energy estimates (which deal with the norm in $L_2(\Omega_R)$ of the difference $\nabla u - \nabla u^R$). Nevertheless, a method of deriving asymptotically accurate estimates of the solutions of singularly perturbed problems in weighted L_2 norms has been developed [6, 9, 18, 19, 20]. This method was transferred in [5] to weighted norms, generated by Hölder classes and L_p. We will formulate one of the results obtained in [5].

Suppose $1 = 1, 2, \ldots$ and $p \in (1, \infty)$ are indicators of smoothness and summability and β and γ are weighting factors which satisfy the inequalities

$$-\frac{1}{2} < \beta - l - \frac{3}{2} + \frac{3}{p} < \frac{1}{2} < \gamma - l - \frac{3}{2} + \frac{3}{p} < \frac{3}{2} \tag{3.5}$$

We will assume that $g = 0$ (for simplicity) and the right-hand of system (1.1) satisfies the relation

$$|||f|||_{l-1,\gamma} = \left(\sum_{k=0}^{l-1} \int_\Omega |x|^{p(\gamma - l + 1 + k)} |\nabla^k f(x)|^p \, dx \right)^{1/p} < \infty$$

Here $\nabla^k f$ is the set of all derivatives of order k of the vector function f. The choice of the operator M^R in the form (1.8), (2.10) then guarantees the following relation between the solutions u and u^R of problems (1.1), (1.2) and (1.4)–(1.6)

$$|||u - u^R|||_{l+1,\beta,R} = \left(\sum_{k=0}^{l+1} \int_{\Omega_R} |x|^{p(\beta - l - 1 + k)} |\nabla^k u(x) - \right.$$

$$\left. -\nabla^k u^R(x)|^p \, dx \right)^{1/p} \leqslant c_{\beta,\gamma} R^{\beta - \gamma} |||f|||_{l-1,\gamma} \tag{3.6}$$

The constant $c_{\beta,\gamma}$ is independent of both f and $R \geqslant R^0$. Estimate (3.6) is asymptotically accurate, i.e. when β and γ reaches one of the limits indicated in (3.5), $c_{\beta,\gamma}$ loses the stated property of independence.

We emphasize that, when conditions (3.5) are satisfied, the difference $\gamma - \beta$, occurring in (3.6), can be made as close as desired to two. If $p = 2$ and $\beta = 1 = 1$, relation (3.6) takes the form of an estimate in an energy metric.

Inequality (3.6), assumption (2.7) and the possibility of constructing a partial sum of series (2.1) imply a pointwise weighted estimate, which, when using the notation employed in (2.7), looks particularly simple. Thus, if relation (2.7) is satisfied, we have

$$u(x) - u^R(x) = O(R^{-2}|x|^0) \tag{3.7}$$

Without dwelling on the physical interpretation of the conditions of rigid clamping of the surface of the internal cavity, we emphasize that all the above results hold for the Dirichlet conditions (1.3). For them formula (3.7) is the final one, but in the case of the Neumann conditions (1.2), when (2.7) is satisfied with $\varepsilon > 1$, the right-hand side of (3.7) may decrease to $O(R^{-3}|x|^1)$ (see Sections 4 and 6 later).

We will not consider further the problem of justifying the asymptotic expansions, since plane problems are solved using the same scheme from [5, 6, 9, 18, 19].

4. THE TWO-DIMENSIONAL PROBLEM WITH BOUNDARY CONDITIONS IN THE STRESSES

Suppose $n = 2$, i.e. Ω and Ω_R are the plane $\mathbf{R}^2$ and the circle B_R respectively with an opening ω^-. If requirement (2.7) holds, the solution u of the external Neumann problem (1.1), (1.2) is determined, apart from a constant term $c \in \mathbf{R}^2$, and allows of the representation

$$u(x)=c+\sum_{j=1}^{6} b_j T^{(j)}(x)+O(|x|^{-2}) \tag{4.1}$$

Here $T^{(1)}$ and $T^{(2)}$ are columns of a 2×2 Boussinesq matrix (a two-dimensional Somigliana tensor) with elements

$$T_{ij}(x)=\alpha\{-\delta_{ij}\ln|x|+\gamma x_i x_j|x|^{-2}\} \tag{4.2}$$

$$\alpha=(\lambda+3\mu)[4\pi\mu(\lambda+\mu)]^{-1},\quad \gamma=(\lambda+\mu)(\lambda+3\mu)^{-1}$$

$$T^{(3)}=\frac{1}{2}(\partial_1 T^{(2)}-\partial_2 T^{(1)}),\quad T^{(4)}=\frac{1}{2}(\partial_1 T^{(2)}+\partial_2 T^{(1)})$$

$$T^{(5)}=\partial_1 T^{(1)},\quad T^{(6)}=\partial_2 T^{(2)},\quad \partial_i=\partial/\partial x_i,\quad i=1,\ 2 \tag{4.3}$$

The coefficients b_1, b_2 and b_3 have the meaning of the components of the principal vector and of the principal moment of the loads, and are found from the formulae

$$b_i=\int_{\Omega} f_i dx+\int_{\partial\omega} g_i ds_x$$

$$b_3=\int_{\Omega}(x_2 f_1-x_1 f_2)dx+\int_{\partial\omega}(x_2 g_1-x_1 g_2)ds_x \tag{4.4}$$

We will first set the Dirichlet conditions on the far boundary ∂B_R and we will seek the asymptotic form of u^R as the partial sum of series (2.1). In order to reduce the discrepancy u in (1.5), we will choose the right-hand side of h^R in accordance with (4.1)

$$u^R(x)=h^R(x)\equiv b_1 T^{(1)}(x)+b_2 T^{(2)}(x),\quad x\in\partial B_R \tag{4.5}$$

We emphasize that b_1 and b_2 are calculated directly from f and g. Now, by (4.1) and (4.3) the discrepancy u in (4.5) is

$$c+\sum_{j=3}^{6} b_j T^{(j)}(x)+O(|x|^{-2-\varepsilon})=c+R^{-1}\sum_{j=3}^{6} b_j T^{(j)}(\xi)+O(R^{-2-\varepsilon})$$

(We assume that $\varepsilon\in(0,1)$ in (2.7).) Hence, $w^0=-c$ while w^1 is the solution of system (2.2) with the boundary condition

$$w^1(\xi)=-\sum_{j=3}^{6} b_j T^{(j)}(\xi),\quad \xi\in\partial B_1 \tag{4.6}$$

Here the smooth vector function $R^{-1}w^1(R^{-1}x)$ generates a discrepancy $O(R^{-2})$ in (1.6) and hence $\upsilon^1=0$ and the term υ^2 of series (2.1) vanishes at infinity as $O(|x|^{-1})$.

We will consider the Neumann problem in Ω_R. To satisfy the solvability conditions we will specify forces on the external contour ∂B_R which balance f and g

$$N(x,\ \nabla)u^R(x)\equiv\sigma^{(\nu)}(u^R;\ x)=\sum_{j=1}^{3} B_j^R\sigma^{(\nu)}(T^{(j)};\ x),\quad x\in\partial B_R \tag{4.7}$$

The quantities b^R are obtained from (4.4), in which the integration set Ω is replaced by Ω_R; by virtue of (2.7) we have

$$|b_1^R-b_1|+|b_2^R-b_2|+R^{-1}|b_3^R-b_3|\leq cR^{-2-\varepsilon}$$

Since b_j^R differs only slightly from b_j, the plan for constructing the initial terms of series (2.1) remains the same as for conditions (4.5), but we can put $w^0=0$ and relation (4.6) must be replaced by

$$\sigma^{(\nu)}(w^1;\ \xi)=-\sum_{j=4}^{6} b_j\sigma^{(\nu)}(T^{(j)};\ \xi),\quad \xi\in\partial B_1 \tag{4.8}$$

Further, we mean by υ^0 the solution u for which $c = 0$ in (4.1), i.e. $w^0 = 0$ always. Taking into account terms $O(|x|^{-2})$ in expansion (4.1) (combinations of the second derivatives of the columns of $T^{(1)}$ and $T^{(2)}$ we can also determine the term $R^{-2}w^2$ in (2.1); it is a vector function that is continuous in $\bar{B}_1$. Thus, in both versions

$$u^R(x) = \upsilon^0(x) + R^{-1}w^1(R^{-1}x) + R^{-2}\upsilon^2(x) + \\ + R^{-2}w^2(R^{-1}x) + O(R^{-2-\varepsilon}|x|^0) \tag{4.9}$$

Since the solution of problem (1.1), (1.2) itself is determined apart from an additive constant, it makes sense to estimate the difference $u - u^R - c^R$, where $c^R \in \mathbf{R}^2$ is any convenient column. By (4.9) when $c^R = c - R^{-1}w^2(0) - R^2w^2(0)$ with $x \in \Omega_R$

$$u(x) - u^R(x) - c^R = O(R^{-2}|x|^1) \tag{4.10}$$

Both problems (1.4), (1.6), (4.5) and (1.4), (1.6), (4.7) give an approximation of the first order for the displacements, and of the second order for the stresses and derivatives of the displacements. The problem with the Dirichlet conditions possesses some advantages: the structure of the right-hand side in (4.5) in simpler compared with (4.7) and it is uniquely solvable. Moreover, the arbitrariness in the choice of the solutions of problems (1.1), (1.2) and (1.4), (1.6), (1.7) is still not the same; in the first case it is the vector $c = (c_1, c_2)$ while in the second it is the vector $c + c_3\theta$, where θ is the rotation $(x_2, -x_1)$.

5. THE TWO-DIMENSIONAL PROBLEM WITH BOUNDARY CONDITIONS IN THE DISPLACEMENTS

As in the previous section, $n = 2$. We know (see, for example, [18, Section 6.4]) that when requirement (2.7) is satisfied there is a unique bounded solution u of problem (1.1), (1.3). It can be represented in the form

$$u(x) = c + \sum_{j=3}^{6} b_j T^{(j)}(x) + O(|x|^{-2}) \tag{5.1}$$

The column $c = (c_1, c_2)$ and the factors $b_3, \ldots, b_6$ depend on f and g, while $T^{(j)}$ are defined in (4.3). In the further asymptotic constructions we will claim a number of special solutions of homogeneous problem (1.1), (1.3) which increase at infinity. Namely

$$\zeta^1 = T^{(1)} + \zeta^{10},\ \zeta^2 = T^{(2)} + \zeta^{20},\ \zeta^3 = \theta + \zeta^{30} \tag{5.2}$$

Here ζ^{i0} and ζ^{30} are bounded energy solutions of problem (1.1), (1.3) with right-hand sides $f = 0$, $g = -T^{(i)}$ and $g = -\theta$ $(i = 1, 2)$. By (6.1) the following expansions hold

$$\zeta^{j0}(x) = (Q_{j1}, Q_{j2}) + \sum_{m=3}^{6} P_{jm} T^{(m)}(x) + O(|x|^{-2}),\ j = 1,2,3 \tag{5.3}$$

The matrix $Q = (Q_{jk})_{j,k=1}^2$ is symmetrical; we will denote its columns by $Q^{(1)}$ and $Q^{(2)}$. Using solutions (5.2) (weighting functions) we calculate the coefficients c_1, c_2 and b_3 from (5.1)

$$\begin{aligned} c_i &= -\int_\Omega f \cdot \zeta^i dx + \int_{\partial\omega} g \cdot \sigma^{(\nu)}(\zeta^i) ds_x,\ i = 1,2 \\ b_3 &= \int_\Omega f \cdot \zeta^3 dx - \int_{\partial\omega} g \cdot \sigma^{(\nu)}(\zeta^3) ds_x \end{aligned} \tag{5.4}$$

All the information on ζ^j can be found in [10].

Finally, using the last equation in (5.4) and recalling that ζ^{30} solves problem (1.1), (1.3) with $f = 0$, $g = -\theta$, we obtain the sign of the coefficient P_{33} in (5.3) with $j = 3$

$$P_{33} = \int_{\partial\omega} \theta \cdot \sigma^{(\nu)}(\zeta^3) ds_x = -\int_{\partial\omega} \zeta^{30} \cdot \sigma^{(\nu)}(\zeta^{30}) ds_x = -2E(\zeta^{30}; \Omega) < 0 \tag{5.5}$$

In (5.5) we have used Betti's identity, and we mean by E the functional of the elastic energy

$$E(u;\Xi)=\frac{1}{2\mu}\sum_{i,j=1}^{2}\int_{\Xi}(\sigma_{ij}(u)^2-\frac{\lambda}{2(\lambda+\mu)}\sigma_{ii}(u)\sigma_{jj}(u)dx \tag{5.6}$$

We will now investigate problem (1.4), (1.5), (1.7) in Ω_R. We specify the Dirichlet condition (2.9) on the far boundary ∂B_R. It is taken to be homogeneous because, according to (5.4), the coefficients c_i in (5.1) are unavailable without preliminary solution of the external problem (1.1), (1.3)—we must know the weighting functions ζ^i. When constructing the asymptotic form, we will use the method proposed in [21] (see also [17, 6, 9]) and we will seek the principal term of series (2.1) for u^R as the linear combination

$$\upsilon^0=u+B_1\zeta^1+B_2\zeta^2 \tag{5.7}$$

We will calculate column B of coefficients B_i in (5.7) when determining the term w^0 of series (2.1), starting from the requirement $w^0(0)=0$ (otherwise, the discrepancy $w^0(0)\neq 0$ of the sum υ^0+w^0 in boundary condition (1.7) is unacceptably high). By (5.1)–(5.3) and (4.2) in the neighbourhood of ∂B_R

$$\upsilon^0(x)=c+\sum_{j=1}^{2}B_j(T^{(j)}(x)+Q^{(j)})+O(R^{-1})=H(\xi)+O(R^{-1})$$

$$H(\xi)=c+\alpha B\ln R+\sum_{j=1}^{2}B_j(T^{(j)}(\xi)+Q^{(j)})$$

Consequently, w^0 is the solution of system (2.2) with boundary condition $w^0(\xi)=-H(\xi)$, $\xi\in\partial B_1$. It can be represented in the form

$$w^0(\xi)=-c-\alpha B\ln R+\sum_{j=1}^{2}B_j(\eta^{j0}(\xi)-Q^{(j)}) \tag{5.8}$$

We have denoted by η^{j0} the solution of the same system, which is identical with $-T^{(j)}$ on ∂B_1. The sum $\eta^j=T^{(j)}+\eta^{j0}$ describes the displacement field in a disc B_1 with a clamped edge acted upon by a unit force in the direction x_j, concentrated at its centre. From the columns $q^{(j)}=\eta^{j0}(0)$ we set up a 2×2 matrix q (it is symmetrical). By virtue of (5.8) the required relation $w^0(0)=0$ is equivalent to the algebraic system

$$\{-\alpha\mathbf{1}\ln R+q-Q\}B=c \tag{5.9}$$

Here $\mathbf{1}$ is the unit 2×2 matrix. The matrix of system (5.9) is non-singular for sufficiently large R.
Hence, from (5.9) we obtain the coefficients $B_j=O(|\ln R|^{-1})$. Moreover,

$$w^0(\xi)=\sum_{j=1}^{2}B_j(\eta^{j0}(\xi)-\eta^{j0}(0))=O(|\ln R|^{-1})$$

Thus, the formulation on ∂B_R of the homogeneous Dirichlet conditions introduces a perturbation $O(|\ln R|^{-1})$ into the principal term υ^0 of series (2.1). It is obvious that $u^R(x)$ converges to $u(x)$ as $R\to\infty$ for any fixed x, but the rate of convergence is extremely slow. We state finally that

$$|u(x)-u^R(x)|\leqslant c_0|\ln R|^{-1}(1+|\ln|x||)$$

$$|\nabla^k u(x)-\nabla^k u^R(x)|\leqslant c_k|\ln R|^{-1}|x|^{-k},\quad k=1,2,\ldots$$

We now turn to problem (1.4), (1.5), (1.7) with conditions in the stresses on ∂B_R (i.e. $M^R=N$). We will first try to take $\upsilon^0=u$. Calculating the discrepancy u in the homogeneous conditions (4.7) on ∂B_R, we conclude that $w^0=0$, while the term w^1 must satisfy system (2.2) and the boundary condition

$$\sigma^{(\nu)}(w^1;\xi)=-\sum_{j=3}^{6}b_j\sigma^{(\nu)}(T^{(j)};\xi),\ \ \xi\in\partial B_1 \tag{5.10}$$

If $b_3=0$, the Neumann problem obtained in B_1 has no continuous solution (the principal moment of the load is not equal to zero). Hence, we must change the structure of the principal term of series (2.1). We will put

$$v^0 = u - b_3 P_{33}^{-1} \zeta^3 \tag{5.11}$$

The vector function v^0 satisfies problem (1.1), (1.3) and, by (5.1)–(5.3) and (5.5) admits of the expansion

$$v^0(x) = \theta(x) + C + \sum_{j=4}^{6} C_j T^{(j)}(x) + O(|x|^{-2}) \tag{5.12}$$

Two facts turn out to be important: the term $T^{(3)}$, which generates a non-zero moment in the load (5.10), vanishes from (5.12), and the operator N annuls the rigid displacement $\theta + C$.

Thus, the boundary condition for w^1 acquires the form (4.8) (with new coefficients), and the corresponding problem becomes solvable. Nevertheless, in view of (5.11) and (2.2) $u(x) - u^R(x) = O(1)$, and problem (1.4), (1.7) with homogeneous condition (4.7) cannot, in general, be regarded as an approximation of problem (1.1), (1.3).

Unsatisfactory results were obtained on setting both stable (Dirichlet) and natural (Neumann) boundary conditions on $\Gamma_R = \partial B_R$; we will attempt to find an appropriate artificial condition. In an asymptotic analysis of the problem the presence of the third condition for the Neumann problem to be solvable in B_1 (related to the moment) turned out to be disastrous. The conditions for solvability are generated by fields on which the energy functional is annulled (this observation is trivial thanks to Betti's identity). A set of Green's formulae exists for the Lamé operator, on the right-hand sides of which there are quadratic forms, which differ from (5.6). We will indicate one of these, namely,

$$\int_{\Xi} u \cdot Lu dx + \int_{\partial\Xi} u \cdot \tau^{(\nu)}(u) ds_x = 2G(u;\Xi) \tag{5.13}$$

$$G(u;\Xi) = \frac{1}{2} \int_{\Xi} (\mu |\nabla u_1|^2 + \mu |\nabla u_2|^2 + (\lambda+\mu)|\nabla \cdot u|^2) dx \tag{5.14}$$

$$\begin{gathered} \tau_{ii}(u) = \mu \frac{\partial u_i}{\partial x_i} + (\lambda+\mu)\nabla \cdot u, \quad \tau_{12}(u) = \mu \frac{\partial u_1}{\partial x_2}, \quad \tau_{21}(u) = \mu \frac{\partial u_2}{\partial x_1} \\ \tau^{(\nu)} = (\tau_1^{(\nu)}, \tau_2^{(\nu)}), \quad \tau_i^{(\nu)} = \nu_1 \tau_{i1} + \nu_2 \tau_{i2} \quad (i = 1,2) \end{gathered} \tag{5.15}$$

The quantities (5.15) and (5.14) have no physical meaning; they can be called quasi-stresses and quasi-energies. However, the fact that $G(u;\ \Xi) = 0$ only for the constant vector u is decisive: now the term w^1 of series (2.1) is a solution of system (2.2) with boundary condition

$$M(\xi, \nabla_\xi) w^1(\xi) \equiv \tau^{(\nu)}(w^1;\xi) = -\sum_{j=3}^{6} b_j \tau^{(\nu)}(T^{(j)};\xi), \quad \xi \in \delta B_1 \tag{5.16}$$

Since the mean of the right-hand side of (5.16) over ∂B_1 is zero, a continuous solution of this problem exists. It is determined, apart from a constant term, and becomes unique for the normalization $w^1(0) = 0$. Using the usual scheme we can convince ourselves that $v^0 = u$, $w^0 = 0$, $v^1 = 0$ in (2.1) (in particular, Eqs (2.5) are true when $K = 1$). Thus, if we define the operator $M^R(x, \nabla)$ by formulae (2.4) with $\varkappa = 1$ and (5.16), problem (1.4), (1.5), (1.7) yields a first-order approximation to the solution of problem (1.1), (1.3), which is undoubtedly better than in the previous cases. Finally, the following relation is satisfied

$$u(x) - u^R(x) = O(R^{-2}|x|^1) \tag{5.17}$$

The formulation of these artificial conditions in order to approximate the external Neumann problem also possesses its own merits: while retaining the accuracy of the approximation of problem (1.4)–(1.6) there is the same arbitrariness (constant vectors) in choosing the solution as in problem (1.1), (1.2).

6. NOTES, COROLLARIES AND GENERALIZATIONS

1. The results obtained in Sections 4 and 5 also hold for an arbitrary (necessarily circular) truncating contour $\Gamma_R = \{x: R^{-1}x \in \Gamma\}$. Moreover, in the case of stable or natural conditions, the elastic material can be anisotropic or even non-uniform, but with different moduli, rapidly stabilizing at infinity. Any non-uniformity hinders the

construction of artificial conditions of the type (5.16) (since Green's formula is fixed). If, in the case of anisotropy, we write Green's formula (5.13) with quasi-energy G, which is only degenerate on constant vectors, then on Γ_R we can formulate the boundary conditions in quasi-stresses.

2. The weighted estimates used in [5] and here best reflect the behaviour of the difference $u(x) - u^R(x)$ in different zones (in the region of $\partial\omega$, when $|x| = O(R)$, etc.). We will illustrate this by comparing the (unamended) relations (3.7) and (5.17) in the case of conditions (1.3) with $n = 3$ and (1.2) with $n = 2$.

Formula (3.7) indicates that in a sphere with a fixed radius (independent of R) the difference $u - u^R$ and all its derivatives do not exceed cR^{-2} in modulus (the factor c depends on the radius and the order of the derivative). The same estimate holds for the difference itself everywhere in Ω_R, but at greater distances (of the order of R) it is improved for the derivatives. In particular, when $|x| > R/2$ (in the region of ∂B_R) by (3.7) we have

$$\left|\nabla^k u(x) - \nabla^k u^R(x)\right| \leq c_k R^{-2} |x|^{-k} \leq C_k R^{-2-k}$$

The accuracy of the approximation around the contour $\partial\omega$, guaranteed by formula (5.17), is $O(R^{-2})$, despite the fact that, on the whole, the approximation is only of the first order—according to (5.17) the difference between $u(x)$ and $u^R(x)$ is $O(R^{-2})$ on $\partial\omega$, and increases as $|x|$ increases to $O(R^{-1})$. Hence, (5.17) contains more accurate information on $|u(x) - u^R(x)|$ than the estimate using the maximum of the modulus (2.6) with $K = 1$ (compare the end of Section 1). Extending the "interpretation" of (5.17), we have

$$\left|\nabla^k u(x) - \nabla^k u^R(x)\right| \leq c_k R^{-2} |x|^{1-k}$$

Thus, for the first derivatives of the displacements (the strains and stresses) the error does not exceed $c_1 R^{-2}$ everywhere in Ω_R, while for the leading derivatives it decreases with distance from $\partial\omega$.

In the integral estimates of the form (3.6) the selection of zones, in which the approximation is characterized by different orders of accuracy, is achieved by varying the weighting factors β and γ. Note also that the diversity of the weighted estimates in Section 4 is due to the need to modify the weighted norms for two-dimensional Neumann problems ([18, Section 6.1]).

3. In a plane region Ω we consider the Neumann problem

$$-\Delta u(x) = f(x), \ x \in \Omega; \ \partial_\nu u(x) = g(x), \ x \in \partial\omega \tag{6.1}$$

We take the following as the approximating problem

$$-\Delta u^R(x) = f(x), \quad x \in \Omega_R = B_R \setminus \overline{\omega}; \quad \partial_\nu u^R(x) = g(x), \ x \in \partial\omega \tag{6.2}$$

$$\partial_\nu u^R(x) + R^{-1} u^R(x) = -b_0 (2\pi R)^{-1} (1 + \ln R), \quad x \in \partial B_R \tag{6.3}$$

Here $\partial_\nu = \nu \cdot \nabla$, and the scalar function f is subject to requirement (2.7) with $\varepsilon > 1$ and

$$b_0 = \int_\Omega f(x)dx + \int_{\partial\omega} g(x) ds_x \tag{6.4}$$

The following expansion holds for the solution u

$$u(x) = b_0 \Phi(x) + c + \sum_{i=1}^{2} b_i \frac{\partial}{\partial x_i} \Phi(x) + O(|x|^{-2}) \tag{6.5}$$

Here Φ is the fundamental solution of (3.4) while c is an arbitrary constant. We fixed $u = \upsilon^0$ so that $c = 0$ in (6.5). The artificial condition (6.3) is characteristic of the fact that when $x \in \partial B_R$

$$\left(\partial_\nu + \frac{1}{R}\right) \frac{\partial}{\partial x_i} \Phi(x) = -\frac{1}{2\pi} \left(\partial_\nu + \frac{1}{R}\right) \frac{x_i}{|x|^2} = 0$$

Consequently, when constructing series (2.1) we find that υ^0 generates in (6.3) a discrepancy $O(R^{-3})$, and hence $w^0 = w^1 = 0$ and $\upsilon^1 = \upsilon^2 = 0$. After taking account of asymptotic terms of the order of $|x|^{-2}$ and $|x|^{-3}$ in (6.4) (combinations of the leading derivatives of Φ) we can determine w^2 and υ^3 and w^3 in (2.1).

Thus, like (4.10) we obtain a constant c^R such that, in the notation of (2.7),

$$u(x) - u^R(x) - c^R = O(R^{-3} |x|) \tag{6.6}$$

An increase in the approximation accuracy has occurred due to the choice of the same operator (3.2) as in the three-dimensional situation, but now it cancels not the principal term of expansion (3.4) but the first term, ignored on the right-hand side of (6.3).

Using an affine transformation, any second-order scalar elliptic operator is transformed into a Laplace operator. Hence, for the external Neumann problem with any such operator, a truncating contour Γ_R and an operator M_R

in it exist which yields an approximation of higher accuracy (6.6). We were unable to construct such a surface and operator for the system of Lamé equations.

4. In the case of the Laplace operator, all the methods from the preceding sections can be transferred without difficulty to corner regions. We will discuss the Dirichlet problem

$$-\Delta u(x) = f(x),\ x \in \Omega;\ u(x) = g(x),\ x \in \partial\Omega \tag{6.7}$$

Here $\Omega \subset \mathbf{R}^2$ is a region with a smooth boundary (for simplicity), which coincides outside the circle B^0_R with the corner $\mathbf{K} = \{x: r > 0,\ \varphi \in (0, \alpha)\}$; r, φ are polar coordinates, and $r = |x|$, $\alpha \in (0, 2\pi)$. We will assume that the carriers f and g are compact (this condition can be replaced by the similar condition (2.7)). A unique bounded solution of problem (6.7) exists; for this we have the expansion

$$u(x) = c_1 r^{-\Lambda}\sin(\Lambda\varphi) + c_2 r^{-2\Lambda}\sin(2\Lambda\varphi) + O(|x|^{-3\Lambda}) \tag{6.8}$$

In (6.8) $\Lambda = \pi/\alpha$, and c_1 and c_2 are constants which depend on f and g. The principal term of expansion (6.8) suggests the form of the artificial conditions on the arc $\Gamma_R = \partial B_R \cap \mathbf{K}$ namely

$$\begin{aligned} &-\Delta u^R(x) = f(x),\quad x \in \Omega_R = \Omega \cap B_R \\ &u^R(x) = g(x),\quad x \in \partial\Omega_R \cap B_R \\ &\partial_\nu u^R(x) + \Lambda R^{-1} u^R(x) = 0,\quad x \in \Gamma_R \end{aligned} \tag{6.9}$$

Using methods developed in [7, 9, 19], the solution u^R can be decomposed into an asymptotic series in inverse powers of the parameter $R^{-\Lambda}$. The initial terms of this series are

$$u^R(x) = u(x) + R^{-2\Lambda} w^2(R^{-1}x) + R^{-3\Lambda} \upsilon^3(x) + \ldots \tag{6.10}$$

The function w^2 is found from the problem in the sector $\mathbf{K}_1 = \mathbf{K} \cap B_1$

$$-\Delta_\xi w^2(\xi) = 0,\ \ \xi \in \mathbf{K}_1;\ \ w^2(\xi) = 0,\ \ \xi \in \partial\mathbf{K} \cap B_1$$

$$\partial_{\nu(\xi)} w^2(\xi) + \Lambda w^2(\xi) = c_2 \Lambda |\xi|^{-2\Lambda}\sin(2\Lambda\varphi),\ \ \xi \in \Gamma_1$$

The representation $w^2(\xi) = b_1|\xi|^\Lambda \sin(\Lambda\varphi) + O(|\xi|^{2\Lambda})$ holds.

We introduce the geometrical constraint $\Omega \subset \mathbf{K}$; it is not fundamental but it abbreviates the formulae (otherwise, by [7, 9, 19], in (6.10) we need to multiply w^2 by the truncating function). In this case υ^3 is the solution of the Dirichlet problem with finite right-hand side

$$-\Delta \upsilon^3(x) = 0,\ \ x \in \Omega;\ \ \upsilon^3(x) = -b_1 r^\Lambda \sin(\Lambda\varphi),\ \ x \in \partial\Omega$$

The relation $\upsilon^3(x) = O(|x|^{-\Lambda})$ is satisfied.

Summing up we have

$$u(x) - u^R(x) = O(R^{-3\Lambda}|x|^\Lambda)$$

For (6.11) to hold, strict requirements are imposed on f and g. They can be reduced by converting to asymptotically accurate estimates, which are obtained by direct specification of the results [7, 9, 19], relating to singular perturbations of the regions close to conical, corner or internal points. In particular, due to the features of the leading derivatives of the solution u^R at corner points (the ends of the arc Γ_R; when $f \neq 0$ and $g \neq 0$ on Γ_R) the estimation is carried out using norms containing additional weighting factors, and powers of the distances to these points. All this relates in equal measure to the Neumann problem in Ω, for which the artificial condition (6.9) takes the form

$$\partial_\nu u^R(x) + \Lambda R^{-1} u^R(x) = (2\alpha R)^{-1} b_0 (1 + \Lambda \ln R),\ \ x \in \Gamma_R$$

It is completely analogous to condition (6.3), where even the factor b_0 is calculated using the previous formula (6.4). In general, the relation between the problems in the corner region and in the exterior of the compact set is closer than appears at first sight.

In fact, by a conformal transformation the corner $\mathbf{K}$ and the region Ω can be transformed into the half-plane $\mathbf{R}^2_+$ and into the region $\mathbf{R}^2_+ \backslash \Upsilon$ respectively, where Υ is a certain compact set; now an evenness (the Neumann condition) or an oddness (the Dirichlet condition) extension converts the problem to a problem in a plane with an opening. These transformations are impossible for an elastic problem in stresses and hence the question of constructing artificial boundary conditions remains open even for the simplest problem of a crack of a normal cleavage in an isotropic plane. However, the asymptotically accurate estimates themselves (for example, for solutions of problems with natural conditions on Γ_R) are obtained by reference to the general results in [7, 9, 19] (see also [22]).

This research was supported financially by the Russian Foundation for Basic Research (96-01-01069).

REFERENCES

1. GOLDSTEIN, C. I., The finite element method with nonuniform mesh sizes for unbounded domains. *Math. Comput.*, 1981, **36**, 154, 387–104.
2. GOLDSTEIN, C. I., Multigrid method for elliptic problems in unbounded domains. *SIAM J. Numer. Anal.*, 1993, **30**, 1, 159–183.
3. GUIRGUIS, G. H., On the coupling boundary integral and finite element methods for the exterior Stokes problem in 3-D. *SIAM J. Numer. Anal.*, 1987, **24**, 2, 310–322.
4. GUIRGUIS, G. H. and GUNZBERGER, M. D., On the approximation of the exterior Stokes problem in three dimensions. *RAIRO Modél. Math, Anal. Numer.* 1987, **21**, 3, 445–464.
5. NAZAROV, S. A. and SPECOVIUS-NEUGEBAUER, M., The approximation of unbounded regions by bounded one. Boundary-value problems for the Lamé operator. *Albegra i Analiz.*, 1996, **8**, 5, 229–268.
6. MAZ'YA, V. G., NAZAROV, S. A. and PLAMENEVSKII, B. A., *The Asymptotic Form of Solutions of Elliptic Boundary-value Problems in the Case of a Singular Perturbation of the Region.* Izd. Tbil. Univ., Tbilisi, 1981.
7. NAZAROV, S. A., *Introduction to Asymptotic Methods of the Theory of Elasticity.* Izd. Leningrad. Gos. Univ., Leningrad, 1983.
8. NAZAROV, S. A. and PAUKSHTO, M. V., *Discrete Problems and Averaging in Problems of the Theory of Elasticity.* Izd. Leningrad. Gos. Univ., Leningrad, 1984.
9. MAZJA, W. G., NASAROV, S. A., PLAMENEWSKI, B. A., *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten.* Berlin: Akademie-Verlag, 1990, 431.
10. ZORIN, I. S. and NAZAROV, S. A., The stress–strain state of an elastic space with a thin toroidal inclusion. *Izv. Akad. Nauk SSSR. MTT*, 1985, 3, 79–86.
11. ZORIN, I. S., MOVCHAN, A. B. and NAZAROV, S. A., The application of the elastic capacity and polarization tensors and the associated deformation, in: *Research on Elasticity and Plasticity.* Izv. Akad. Nauk SSSR. MTT, 1988, 6, 128–134.
12. ZORIN, I. S., MOVCHAN, A. B. and NAZAROV, S. A., The application of the elastic capacity and polarization tensors and the associated deformation, in: *Research on Elasticity and Plasticity.* Izd. Leningrad. Gos. Univ., Leningrad, 1990, 16, 75–91.
13. NAZAROV, S. A., The elastic capacity and polarization of a defect in an elastic layer. *Izv. Akad. Nauk SSSR. MTT*, 1990, 5, 57–65.
14. PÓLYA, G. and SZEGÖ, G., *Isoperimetric Inequalities in Mathematical Physics.* Princeton University Press, Princeton, NJ, 1951.
15. MOVCHAN, A. B., Wiener polarization and capacity matrices for the operator of the theory of elasticity in doubly connected regions. *Mat. Zametki*, 1990, **47**, 2, 151–153.
16. MOVCHAN, A. B., Integral characteristics of elastic inclusions and cavities in the two-dimensional theory of elasticity. *Eur. J. Appl. Math.*, 1992, **3**, 1, 21–30.
17. IL'IN, A. M., *Matching of the Asymptotic Expansions of the Solutions of Boundary-value Problems.* Nauka, Moscow, 1989.
18. NAZAROV, S. A., Elliptic problems in domains with piecewise smooth boundaries. Walter de Gruyter, Berlin, 1994, 525.
19. MAZ'YA, V. G., NAZAROV, S. A. and PLAMENEVSKII, B. A., The asymptotic form of solutions of elliptic boundary-value problems for irregular perturbation of a region. In *Problems of Mathematical Analysis.* Izd. Leningrad. Gos. Univ., Leningrad, 1981, 8, 72–153.
20. NAZAROV, S. A., Interaction of concentrated masses in a harmonically oscillating spatial body with Neumann boundary conditions. *RAIRO Modél. Nath. Anal. Numer.*, 1993, **27**, 6, 777–799.
21. IL'IN, A. M., The boundary-value problem for a second-order elliptic equation in a region with a narrow slit. 2. A region with a small opening. *Mat. Sbornik*, 1977, **103**, 2, 265–284.
22. MAZ'YA, V. G. and NAZAROV, S. A., Asymptotic form of the energy integrals for small perturbations of the boundary in the region of corner and conical points. *Trudy Mosk. Mat. Obshchestva*, 1987, **50**, 79–129.

Translated by R.C.G.